SECTION 8.1: REVIEW OF INTEGRATION

RECALL: What is $\int f(x)\,dx$ asking for? How do you check your answer?

BASIC INTEGRATION FORMULAS:

- $\int du = \int 1\,du = u + C$
- $\int u^p\,du = \frac{1}{p+1}u^{p+1} + C, \quad p \neq -1$
- $\int \sin(u)\,du = -\cos(u) + C$
- $\int \cos(u)\,du = \sin(u) + C$
- $\int \csc(u)\cot(u)\,du = -\csc(u) + C$
- $\int \sec(u)\tan(u)\,du = \sec(u) + C$
- $\int \csc^2(u)\,du = -\cot(u) + C$
- $\int \sec^2(u)\,du = \tan(u) + C$
- $\int \frac{1}{u}\,du = \ln|u| + C$
- $\int e^u\,du = e^u + C$

RECALL: If $a > 0$:

- $\int \frac{1}{\sqrt{a^2-u^2}}\,du = \arcsin\left(\frac{u}{a}\right) + C = \sin^{-1}\left(\frac{u}{a}\right) + C$
- $\int \frac{1}{u^2+a^2}\,du = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$
- $\int \frac{1}{u\sqrt{u^2-a^2}}\,du = \frac{1}{a}\operatorname{arcsec}\left(\frac{|u|}{a}\right) + C = \frac{1}{a}\sec^{-1}\left(\frac{|u|}{a}\right) + C$

BASIC INTEGRATION PROPERTIES:

Suppose F and G are antiderivatives of f and g, respectively, on some open interval I.

- **CONSTANT MULTIPLE RULE:** $\int k \cdot f(x)\,dx = k\int f(x)\,dx = k \cdot F(x) + C$
- **SUM AND DIFFERENCE RULE:** $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx = F(x) \pm G(x) + C$
- **LINEAR SUBSTITUTION:** If $a \neq 0$, then $\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + C$.

  For example, $\int \cos(x)\,dx = \sin(x) + C$, so $\int \cos(117x-42)\,dx = \frac{1}{117}\sin(117x-42) + C$.

BASIC STRATEGIES:

- Algebraic manipulations: distribute, 'separate numerators,' rewrite radicals as powers, use properties of exponents, long division, complete the square ...
- Substitution: let u be: what's in parentheses, what's in the exponent, what's in the denominator, ...

EXAMPLE 1 (VIDEO): Find the following antiderivatives (indefinite integrals). Check your answers.

1. $\displaystyle\int \left[x^2 - \cos(x)\right]\, dx$ Ans: $\dfrac{1}{3}x^3 - \sin(x) + C$

2. $\displaystyle\int \sqrt{u}\,\left(u^2 - \sqrt[3]{u}\right)\, du$ Ans: $\dfrac{2}{7}\,u^{\frac{7}{2}} - \dfrac{6}{11}\,u^{\frac{11}{6}} + C$

3. $\displaystyle\int \frac{x^3 - x}{x^2}\, dx$ Ans: $\dfrac{1}{2}x^2 - \ln|x| + C$

4. $\displaystyle\int \frac{x^3 - x}{x^2 + 1}\, dx$ Ans: $\dfrac{1}{2}x^2 - \ln\left(x^2 + 1\right) + C$

5. $\displaystyle\int \frac{1 - \sin(t)}{\cos^2(t)}\, dt$ Ans: $\tan(t) - \sec(t) + C$

6. $\displaystyle\int \frac{\cos^2(t)}{1 - \sin(t)}\, dt$ Ans: $t - \cos(t) + C$

7. $\displaystyle\int \frac{1}{\sqrt{25 - 9x^2}}\, dx$ Ans: $\dfrac{1}{3}\sin^{-1}\left(\dfrac{3x}{5}\right) + C$

8. $\displaystyle\int \frac{1}{x\sqrt{9x^2 - 25}}\, dx$ Ans: $\dfrac{1}{5}\sec^{-1}\left(\dfrac{3}{5}|x|\right) + C$

9. $\displaystyle\int \frac{1}{x^2 - 2x + 10}\, dx$ Ans: $\dfrac{1}{3}\tan^{-1}\left(\dfrac{x - 1}{3}\right) + C$

10. $\displaystyle\int \frac{1}{\sqrt{-9x^2 + 36x - 11}}\, dx$ Ans: $\dfrac{1}{3}\sin^{-1}\left(\dfrac{3(x - 2)}{5}\right) + C$

EXAMPLE 2 (VIDEO): Find the following antiderivatives (indefinite integrals). Check your answers.

1. $\displaystyle\int 2x\sqrt{x^2 + 1}\, dx$ Ans: $\dfrac{2}{3}\left(x^2 + 1\right)^{\frac{3}{2}} + C$

2. $\displaystyle\int \cos(\theta)\sqrt{3\sin(\theta) + 2}\, dx$ Ans: $\dfrac{2}{9}\left(3\sin(\theta) + 2\right)^{\frac{3}{2}} + C$

3. $\displaystyle\int \frac{\sqrt{\ln(x) + 3}}{x}\, dx$ Ans: $\dfrac{2}{3}\left(\ln(x) + 3\right)^{\frac{3}{2}} + C$

4. $\displaystyle\int \frac{e^{2t}}{\left(e^{2t} - 1\right)^{-\frac{1}{2}}}\, dt$ Ans: $\dfrac{1}{3}\left(e^{2t} - 1\right)^{\frac{3}{2}} + C$

EXAMPLE 3 (VIDEO): Find the following antiderivatives (indefinite integrals). Check your answers.

1. $\displaystyle\int \left(1 - \sin^2(t)\right) \sin^3(t) \cos(t)\, dt$ Ans: $\dfrac{1}{4}\sin^4(t) - \dfrac{1}{6}\sin^6(t) + C$

2. $\displaystyle\int \tan^3(\theta) \sec^2(\theta)\, d\theta$ Ans: $\dfrac{1}{4}\tan^4(\theta) + C$

3. $\displaystyle\int x\sqrt{1-2x}\, dx$ Ans: $\dfrac{1}{10}(1-2x)^{\frac{5}{2}} - \dfrac{1}{6}(1-2x)^{\frac{3}{2}} + C$

4. $\displaystyle\int \frac{1}{e^{3x} + e^{-3x}}\, dx$ Ans: $\dfrac{1}{3}\tan^{-1}\left(e^{3x}\right) + C$

THE FUNDAMENTAL THEOREM OF CALCULUS (FToC): If f is **continuous** on the interval $[a, b]$ then:

- $\displaystyle\int_a^b f(x)\, dx = F(x)\Big|_{x=a}^{x=b} = F(b) - F(a)$ where F is **any** antiderivative of f on $[a, b]$.

- $F(x) = \displaystyle\int_a^x f(t)\, dt$ is an antiderivative of f on $[a, b]$. That is, $D_x\left[\displaystyle\int_a^x f(t)\, dt\right] = f(x)$ for $a \leq x \leq b$.

EXAMPLE 4 (VIDEO): Explain why the FToC applies to $\displaystyle\int_1^3 \frac{1}{(1-2x)^2}\, dx$ and evaluate. Ans: $\dfrac{2}{5}$

EXAMPLE 5 (VIDEO):

Explain why $\displaystyle\int_a^b \cos(\omega x + \phi)\, dx = 0$ and $\displaystyle\int_a^b \sin(\omega x + \phi)\, dx = 0$ if $b - a = \dfrac{2\pi}{\omega} k$ for some integer k.

EXAMPLE 6 (VIDEO): Derive the following integration formulas:

1. $\displaystyle\int \tan(u)\, du = -\ln|\cos(u)| + C = \ln|\sec(u)| + C$

2. $\displaystyle\int \sec(u)\, du = \ln|\sec(u) + \tan(u)| + C = -\ln|\sec(u) - \tan(u)| + C$

   HINT: Pythagorean Conjugates . . .

HOMEWORK: Section 8.1: 7 - 71 odd

ADDITIONAL PRACTICE:

Find the following integrals. Yep, you guessed it, check your answer by taking the derivative!

1. $\displaystyle\int \sqrt{4-3x}\,dx$
2. $\displaystyle\int x\sqrt{4-3x}\,dx$
3. $\displaystyle\int \frac{x}{\sqrt{4-9x^2}}\,dx$
4. $\displaystyle\int \frac{1}{\sqrt{4-9x^2}}\,dx$
5. $\displaystyle\int \frac{x^3}{\sqrt{9x^2-4}}\,dx$
6. $\displaystyle\int \frac{1}{x\sqrt{9x^2-4}}\,dx$
7. $\displaystyle\int \frac{x}{9x^2+4}\,dx$
8. $\displaystyle\int \frac{1}{9x^2+4}\,dx$
9. $\displaystyle\int \frac{1}{x^2-6x+9}\,dx$
10. $\displaystyle\int \frac{1}{x^2-6x+10}\,dx$
11. $\displaystyle\int \frac{1}{x+\sqrt{x}}\,dx$
12. $\displaystyle\int \frac{1}{\sqrt{x}+x\sqrt{x}}\,dx$
13. $\displaystyle\int \frac{e^x+1}{e^x}\,dx$
14. $\displaystyle\int \frac{e^x}{1+e^x}\,dx$
15. $\displaystyle\int \frac{e^x}{1+e^{2x}}\,dx$
16. $\displaystyle\int \frac{e^x}{\sqrt{1-e^x}}\,dx$
17. $\displaystyle\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$
18. $\displaystyle\int \frac{1}{\sqrt{e^{2x}-1}}\,dx$
19. $\displaystyle\int \frac{\sin(x)+1}{\cos(x)}\,dx$
20. $\displaystyle\int \frac{\cos(x)}{\sin(x)+1}\,dx$
21. $\displaystyle\int \frac{1}{\sin(x)+1}\,dx$
22. $\displaystyle\int \frac{\arcsin(x)}{\sqrt{1-x^2}}\,dx$
23. $\displaystyle\int \frac{\sqrt{1+\ln(x)}}{x}\,dx$
24. $\displaystyle\int \frac{2^x}{\sqrt{9-4^x}}\,dx$
25. $\displaystyle\int \frac{e^{\arctan(x)}}{x^2+1}\,dx$
26. $\displaystyle\int \frac{1}{x\ln(x)\sqrt{(\ln(x))^2-1}}\,dx$
27. $\displaystyle\int 3e^{2\ln(x)}\,dx$
28. Find $\displaystyle\int \frac{2x^3-4x^2+10x-1}{x^2-2x+5}\,dx$
29. Find $\displaystyle\int \frac{1}{x\sqrt{x-1}}\,dx$ two ways: using $u=\sqrt{x-1}$ and $u=\sqrt{x}$.
30. Evaluate $\displaystyle\int_0^{\pi/2} \sqrt{1-\cos(\theta)}\,d\theta$

CHALLENGE: Find $\displaystyle\int_0^1 \frac{x^4(1-x)^4}{x^2+1}\,dx$

HINTS:

1. Power rule
2. $u = 4 - 3x$, $x = \dfrac{4-u}{3}$
3. $u = 4 - 9x^2$
4. arcsine form
5. $u = 9x^2 - 4$, $x^2 = \dfrac{u+4}{9}$
6. arcsecant form
7. $u = 9x^2 + 4$
8. arctangent form
9. Factor; power rule
10. Complete the square; arctangent form
11. Factor $\sqrt{x}$ from denominator; $u = 1 + \sqrt{x}$.
12. Factor $\sqrt{x}$ from denominator; $u = \sqrt{x}$; arctangent form
13. separate numerator / bring up e^x from denominator as e^{-x} and distribute
14. $u = e^x + 1$
15. $u = e^x$; arctangent form
16. $u = 1 - e^x$
17. $u = e^x$; arcsine form
18. $u = e^x$; arcsecant form (multiply numerator and denominator by e^x)
19. Separate numerator / bring $\cos(x)$ up as $\sec(x)$ / rewrite integrand as $\tan(x) + \sec(x)$.
20. $u = \sin(x) + 1$
21. Multiply numerator and denominator by conjugate: $1 - \sin(x)$ and separate numerator.
22. $u = \arcsin(x)$
23. $u = 1 + \ln(x)$
24. $u = 2^x$; arcsine form
25. $u = \arctan(x)$
26. $u = \ln(x)$; arcsecant form
27. use properties of logs and exponents to reduce integrand to $3x^2$.
28. long division then complete the square.
29. $u = \sqrt{x-1}$ results in an arctangent form; $u = \sqrt{x}$ results in an arcsecant form.
30. $2\sqrt{2} - 2$